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On the Non-Homogeneous Equation of the Eighth Degree with Six Unknowns $x^5-y^5+(x^3-y^3)xy = p(z^2-w^2)^2T^3$

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Abstract

We obtain infinitely many non-zero integer sextuples (x, y, z, w, p, T) satisfying the non-homogeneous equation of degree eight with six unknowns given by $x^5 - y^5 + (x^3 - y^3)xy = p(z^2 - w^2)^2 T^3$. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers, Star numbers, Stella Octangular numbers, Octahedral numbers, Pronic number, Jacobsthal number, Jacobsthal-Lucas number, keynea number, Centered pyramidal numbers are exhibited.

Keywords: Replica Non-homogeneous equation, integral solutions, polygonal numbers, Pyramidal numbers, Centered pyramidal numbers.

MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

SO_n -Stella octangular number of rank n

S_n -Star number of rank n

PR_n - Pronic number of rank n

OH_n - Octahedral number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

$CP_{n,3}$ - Centered Triangular pyramidal number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$CP_{n,7}$ - Centered heptagonal pyramidal number of rank n

Introduction

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1-3].Particularly in [4,5] special equations of sixth degree with four and five unknowns are studied. In [6-8] heptic equations with three and five unknowns are analysed. This paper concerns with the problem of determining non-trivial integral solution of the non- homogeneous equation of eighth degree with six unknowns given by $x^5-y^5+(x^3-y^3)xy=p(z^2-w^2)^2T^3$. A few relations between the solutions and the special numbers are presented.

Method of Analysis

The Diophantine equation representing the non- homogeneous equation of degree eight is given by

$$x^5-y^5+(x^3-y^3)xy=p(z^2-w^2)^2T^3 \quad (1)$$

Introduction of the transformations

$$x=u+v,\ y=u-v,\ z=u+1,\ w=u-1,\ p=v,\quad v>1 \quad (2)$$

in (1) leads to

$$u^2+v^2=T^3 \quad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1)

Approach1:

Let $T=a^2+b^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define

$$(u+iv)=(a+ib)^3 \quad (5)$$

Equating real and imaginary parts in (5) we get

$$\left.\begin{aligned} u&=a^3-3ab^2\\ v&=3a^2b-b^3 \end{aligned}\right\} \quad (6)$$

In view of (2), (4) and (6), the corresponding values of x,y,z,w,p,T are represented by

$$\left.\begin{aligned} x&=a^3+3a^2b-3ab^2-b^3\\ y&=a^3-3a^2b-3ab^2+b^3\\ z&=a^3-3ab^2+1\\ w&=a^3-3ab^2-1\\ p&=3a^2b-b^3\\ T&=a^2+b^2 \end{aligned}\right\} \quad (7)$$

The above values of x,y,z,w,p and T satisfies the following properties:

1. $x(a,1)+y(a,1)+T(a,1)-6p_a^4+6T_{3,a}+12T_{4,a}-8T_{5,a}-2p_a^5+CP_{a,6}=1$
2. $z(a,1)+w(a,1)+T(a,1)-6p_a^4+S_a-6T_{3,a}-19T_{4,a}+16PR_a=0$
3. The following are nasty numbers:
   a) $x(2^{2n},2^{2n})+y(2^{2n},2^{2n})+z(2^{2n},2^{2n})+w(2^{2n},2^{2n})+p(2^{2n},2^{2n})+j_{6n}$
   b) $30(p(2^{2n},2^{2n})+T(2^{2n},2^{2n})-3J_{6n+1}-2KY_{2n}+4j_{2n})$
4. $9(z-w-zw+xy+p^2)$ is a cubic integer.
5. $8\,[2z^2w^2+(x+y)(z+w)(1-p^2)+2p^4-2x^2y^2]$ is a biquadratic integer
6. $(x^2-y^2)(z+w)-8p(zw+1)=0$
7. $zw(z+w)=(x+y)(xy+p^2-1)$

8. $x(a,a)+y(a,a)+z(a,a)+w(a,a)+p(a,a)+T(a,a)+36p_a^3-40T_{3,a}+16p_a^5-8CP_{a,6}=0$

9. $y(a,a)+z(a,a)-w(a,a)+p(a,a)+T(a,a)-SO_a-4T_{3,a}+6(OH_a)-2p_a^5+6CP_{a,6}=0(\text{mod}\,2)$

10. $x^2+y^2-2zw-2p^2=2$

Approach2:

Now, rewrite (3) as,

$$u^2+v^2=T^3\times 1 \quad (8)$$

Also 1 can be written as

$$1=(-i)^n(i)^n \quad (9)$$

Substituting (4) and (9) in (8) and using the method of factorisation, define,

$$(u+iv)=i^n(a+ib)^3 \quad (10)$$

Equating real and imaginary parts in (10) we get

$$\left.\begin{aligned} u&=\cos\frac{n\pi}{2}(a^3-3ab^2)-\sin\frac{n\pi}{2}(3a^2b-b^3)\\ v&=\cos\frac{n\pi}{2}(3a^2b-b^3)+\sin\frac{n\pi}{2}(a^3-3ab^2)\end{aligned}\right\} \quad (11)$$

In view of (2), (4) and (11), the corresponding values of x,y,z,w,p,T are represented

$$\left.\begin{aligned} x&=\cos\frac{n\pi}{2}(a^3-3ab^2+3a^2b-b^3)+\sin\frac{n\pi}{2}(a^3-3ab^2-3a^2b+b^3)\\ y&=\cos\frac{n\pi}{2}(a^3-3ab^2-3a^2b+b^3)-\sin\frac{n\pi}{2}(a^3-3ab^2+3a^2b-b^3)\\ z&=\cos\frac{n\pi}{2}(a^3-3ab^2)-\sin\frac{n\pi}{2}(3a^2b-b^3)+1\\ w&=\cos\frac{n\pi}{2}(a^3-3ab^2)-\sin\frac{n\pi}{2}(3a^2b-b^3)-1\\ p&=\cos\frac{n\pi}{2}(3a^2b-b^3)+\sin\frac{n\pi}{2}(a^3-3ab^2)\\ T&=a^2+b^2\end{aligned}\right\} \quad (12)$$

Approach3:

1 can also be written as

$$1=\frac{((m^2-n^2)+i2mn)((m^2-n^2)-i2mn)}{(m^2+n^2)^2} \quad (13)$$

Following the same procedure as above we get the integral solution of (1) as

$$\left.\begin{aligned} x&=(m^2+n^2)^2[(m^2-n^2)(a^3-3ab^2+3a^2b-b^3)+2mn(a^3-3ab^2-3a^2b+b^3)]\\ y&=(m^2+n^2)^2[(m^2-n^2)(a^3-3ab^2-3a^2b+b^3)-2mn(a^3-3ab^2+3a^2b-b^3)]\\ z&=(m^2+n^2)^2[(m^2-n^2)(a^3-3ab^2)-2mn(3a^2b-b^3)]+1\\ w&=(m^2+n^2)^2[(m^2-n^2)(a^3-3ab^2)-2mn(3a^2b-b^3)]-1\\ p&=(m^2+n^2)^2[(m^2-n^2)(3a^2b-b^3)+2mn(a^3-3ab^2)]\\ T&=(m^2+n^2)^2(a^2+b^2)\end{aligned}\right\} \quad (14)$$

Approach4:
Writing 1 as

$$1=\frac{(2mn+i(m^2-n^2))(2mn-i(m^2-n^2))}{(m^2+n^2)^2}$$

Following the same procedure as above we get the integral solution of (1) as

$$\left.\begin{aligned}
x&=(m^2+n^2)^2[2mn(a^3-3ab^2+3a^2b-b^3)+(m^2-n^2)(a^3-3ab^2-3a^2b+b^3)]\\
y&=(m^2+n^2)^2[2mn(a^3-3ab^2-3a^2b+b^3)-(m^2-n^2)(a^3-3ab^2+3a^2b-b^3)]\\
z&=(m^2+n^2)^2[2mn(a^3-3ab^2)-(m^2-n^2)(3a^2b-b^3)]+1\\
w&=(m^2+n^2)^2[2mn(a^3-3ab^2)-(m^2-n^2)(3a^2b-b^3)]-1\\
p&=(m^2+n^2)^2[2mn(3a^2b-b^3)+(m^2-n^2)(a^3-3ab^2)]\\
T&=(m^2+n^2)^2(a^2+b^2)
\end{aligned}\right\}\quad(15)$$

Approach5:
The solution of (3) can also be obtained as

$$u=m(m^2+n^2),v=n(m^2+n^2),T=(m+n^2)\qquad(16)$$

In view of (16) and (2), the integral solutions of (1) is obtained as

$$\left.\begin{aligned}
x&=(m^2+n^2)(m+n)\\
y&=(m^2+n^2)(m-n)\\
z&=m(m^2+n^2)+1\\
w&=m(m^2+n^2)-1\\
p&=n(m^2+n^2)\\
T&=(m^2+n^2)
\end{aligned}\right\}\quad(17)$$

The above values of x,y,z,w,p and T satisfies the following properties:

1. $z(a,1)+w(a,1)+p(a,1)-4CP_{a,3}-T_{4,a}=1$
2. $6(p(2^{2n},2^{2n})+T(2^{2n},2^{2n})+z(2^{2n},2^{2n})-w(2^{2n},2^{2n})-j_{6n+1}-j_{4n+1})$ is a nasty number.
3. $x(a,1)+y(a,1)+p(a,1)-4P_a^5+2T_{3,a}-5T_{4,a}+2T_{7,a}=1$
4. $4(p(2^{2n+1},2^{2n+1})+T(2^{2n+1},2^{2n+1})-2J_{6n+3}-2KY_{2n+1}+6j_{2n+1})$ is a cubic integer.
5. The following are biquadratic integers:

a)

$$8(p(a^2,1)+T(a^2,1)-48F_{4,a,3}+24CP_{a,3}+22T_{4,a})\ \alpha^3=r^2+s^2,\ \mathrm{u}=\mathrm{r}^2-\mathrm{s}^2,\ \mathrm{v}=2\mathrm{rs},\ \mathrm{r}>\mathrm{s}>0$$

b) $8\,[x(a,a)+y(a,a)+z(a,a)-w(a,a)+p(a,a)-18p_a^4+9GN_a-12T_{3,a}+6T_{4,a}]$

Approach6:

Assuming $T=\alpha^2$ (18)
in (3), we have

$$u^2+v^2=(\alpha^3)^2$$

which is in the form of Pythagorean equation, whose solution is,

$$\alpha^3 = r^2 + s^2,\ \ u = 2rs,\ \ v = r^2 - s^2\ ,\qquad r > s > 0 \qquad \text{(Or)} \qquad (19)$$

$$\alpha^3 = r^2 + s^2,\ u = r^2 - s^2,\ \ v = 2rs \qquad r > s > 0 \qquad (20)$$

Solving the first equation of (19) we have two choices of solutions, namely,

$$r = m(m^2 + n^2), s = n(m^2 + n^2), \alpha = m^2 + n^2,\ m > n > 0 \qquad (21)$$

$$r = m^3 - 3mn^2, s = 3m^2 n - n^3, \alpha = m^2 + n^2,\ m > n > 0 \qquad (22)$$

In view of (18), (19), and (21) and (2), we get the integral solution of (1) as

$$\left.\begin{aligned} x &= (m^2+n^2)^2(2mn + (m^2-n^2)) \\ y &= (m^2+n^2)^2(2mn - (m^2-n^2)) \\ z &= 2mn(m^2+n^2)^2 + 1 \\ w &= 2mn(m^2+n^2)^2 - 1 \\ p &= (m^2+n^2)^2(m^2-n^2) \\ T &= (m^2+n^2)^2 \end{aligned}\right\} \qquad (23)$$

In view of (18), (19), (22) and (2), we get a different integral solution of (1) as

$$\left.\begin{aligned} x &= 2(m^3-3mn^2)(3m^2n-n^3) + m^6 - n^6 + 15m^2n^2(n^2-m^2) \\ y &= 2(m^3-3mn^2)(3m^2n-n^3) - m^6 + n^6 - 15m^2n^2(n^2-m^2) \\ z &= 2(m^3-3mn^2)(3m^2n-n^3) + 1 \\ w &= 2(m^3-3mn^2)(3m^2n-n^3) - 1 \\ p &= m^6 - n^6 + 15m^2n^2(n^2-m^2) \\ T &= (m^2+n^2)^2 \end{aligned}\right\} \qquad (24)$$

Similarly taking (20), instead of (19) and performing the same procedure we will get two more patterns.

Approach7:

Assuming $u = UT,\ v = VT$ (25)

in (3),we get, $U^2 + V^2 = T$ (26)

Assume in (26), $T = w^n\ ,\ w = a^2 + b^2$ (27)

And Write (26) as

$$U^2 + V^2 = w^n \times 1 \qquad (28)$$

Also Write 1 as

$$1 = (-i)^m (i)^m \qquad (29)$$

Substituting (27) and (29) in (28) and using the method of factorisation define

$$(U + iV) = i^m (a + ib)^n \qquad (30)$$

$$= r^n e^{i(m\frac{\pi}{2} + n\theta)},\ \ where\ r = \sqrt{a^2 + b^2},\ \ \theta = \tan^{-1}\frac{b}{a}$$

Equating the real and imaginary parts, we have,

$$\left.\begin{aligned} U &= r^n \cos(m\frac{\pi}{2} + n\theta) \\ V &= r^n \sin(m\frac{\pi}{2} + n\theta) \end{aligned}\right\} \qquad (31)$$

In view of (31), (25) and (2), we get

$$\left.\begin{aligned} x &= r^n[\cos(\frac{m\pi}{2}+n\theta)+\sin(\frac{m\pi}{2}+n\theta)](a^2+b^2)^n \\ y &= r^n[\cos(\frac{m\pi}{2}+n\theta)-\sin(\frac{m\pi}{2}+n\theta)](a^2+b^2)^n \\ z &= r^n[\cos(\frac{m\pi}{2}+n\theta)(a^2+b^2)^n]+1 \\ w &= r^n[\cos(\frac{m\pi}{2}+n\theta)(a^2+b^2)^n]-1 \\ p &= r^n \sin(\frac{m\pi}{2}+n\theta)](a^2+b^2)^n \\ T &= (a^2+b^2)^n \end{aligned}\right\} \quad (32)$$

Conclusion

In conclusion, one may search for different patterns of solutions to (1) and their **c**orresponding properties.

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